

ON THE UNSPLITTABLE MINIMAL ZERO-SUM SEQUENCES OVER FINITE CYCLIC GROUPS OF PRIME ORDER

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ABSTRACT. Let $p > 155$ be a prime and let G be a cyclic group of order p . Let S be a minimal zero-sum sequence with elements over G , i.e., the sum of elements in S is zero, but no proper nontrivial subsequence of S has sum zero. We call S is unsplittable, if there do not exist g in S and $x, y \in G$ such that $g = x + y$ and $Sg^{-1}xy$ is also a minimal zero-sum sequence. In this paper we show that if S is an unsplittable minimal zero-sum sequence of length $|S| = \frac{p-1}{2}$, then $S = g^{\frac{p-11}{2}}(\frac{p+3}{2}g)^4(\frac{p-1}{2}g)$ or $g^{\frac{p-7}{2}}(\frac{p+5}{2}g)^2(\frac{p-3}{2}g)$. Furthermore, if S is a minimal zero-sum sequence with $|S| \geq \frac{p-1}{2}$, then $\text{ind}(S) \leq 2$.

1. INTRODUCTION AND MAIN RESULTS

Let G be a finite abelian group. The Davenport constant $D(G)$ is the smallest integer $\ell \in \mathbb{N}$ such that every sequence S over G of length $|S| \geq \ell$ has a zero-sum subsequence. The studies of the Davenport constant – together with the famous Erdős-Ginzburg-Ziv Theorem – is considered as a starting point in zero-sum theory, and it has initiated a huge variety of further research (more information can be found in the surveys [2, 6, 11], for recent progress see [8, 12, 14, 28]).

The associated inverse problem of Davenport constant studies for the structure of sequences of length strictly smaller than $D(G)$ which do not have a zero-sum subsequence. The index of a sequence is a crucial invariant in the investigation of (minimal) zero-sum sequences (resp. of zero-sum free sequences) over cyclic groups. Recall that the index of a sequence S over G is defined as follows.

Definition 1.1. [11, Definition 5.1.1]

1. Let $g \in G$ be a non-zero element with $\text{ord}(g) = n < \infty$. For a sequence $S = (x_1g) \cdot \dots \cdot (x_lg)$ over G , where $l \in \mathbb{N}_0$ and $1 \leq x_1, \dots, x_l \leq n$, we define $\|S\|_g = \frac{x_1 + \dots + x_l}{n}$ to be the g -norm of S .
2. Let S be a sequence for which $\langle \text{supp}(S) \rangle \subset G$ is a nontrivial finite cyclic group. Then we call $\text{ind}(S) = \min\{\|S\|_g \mid g \in G \text{ with } \langle \text{supp}(S) \rangle = \langle g \rangle\}$ the *index* of S .
3. Let G be a finite cyclic group. $\text{l}(G)$ denotes the smallest integer $l \in \mathbb{N}$ such that every minimal zero-sum sequence S of length $|S| \geq l$ has $\text{ind}(S) = 1$.

Clearly, S has sum zero if and only if $\text{ind}(S)$ is an integer. There are also slightly different definitions of the index in the literature, but they are all equivalent (see Lemma 5.1.2 in [11]).

2000 *Mathematics Subject Classification.* Primary 11B50, 11P70, 20K01.

Key words and phrases: minimal zero-sum sequence, index of sequences.

The index of a sequence was named by Chapman, Freeze and Smith [3]. It was first addressed by Kleitman-Lemke (in the conjecture [15, page 344]), used as a key tool by Geroldinger ([10, page 736]), and then investigated by Gao [5] in a systematical way. Since then it has received a great deal of attention (see for examples [4, 7, 9, 16, 17, 18, 19, 21, 22, 23, 24, 25, 29]).

To investigate the index of long minimal zero-sum sequences, Gao [5] introduced the invariant $\mathsf{l}(G)$. The precise value of $\mathsf{l}(G)$ has been determined independently by Savchev and Chen [20], and by Yuan [27] in 2007.

Theorem 1.2. [20, 27] *Let G be a finite cyclic group of order n . Then $\mathsf{l}(G) = 1$ if $n \in \{1, 2, 3, 4, 5, 7\}$, $\mathsf{l}(G) = 5$ if $n = 6$, and $\mathsf{l}(G) = \lfloor \frac{n}{2} \rfloor + 2$ if $n \geq 8$.*

Let S be a minimal zero-sum (resp. zero-sum free) sequence of elements over an abelian group G . We say that S is *splittable* if there exists an element $g \in \text{supp}(S)$ and two elements $x, y \in G$ such that $x + y = g$ and $Sg^{-1}xy$ is a minimal zero-sum (resp. zero-sum free) sequence as well; otherwise we say that S is *unsplittable*.

Let S be a minimal zero-sum sequence of length $\mathsf{l}(G) - 1$ over a finite cyclic group G . If S is splittable, it is easy to check that $\text{ind}(S) = 1$. If S is unsplittable, Gao [5] conjectured that $\text{ind}(S) = 2$. In 2010, Xia and Yuan [26] showed that Gao's conjecture is true when n is odd, and false when n is even.

Theorem 1.3. [26, Theorem 3.1] *Let S be an unsplittable minimal zero-sum sequence of length $|S| = \mathsf{l}(G) - 1$ over a finite cyclic group G . We have:*

- (1) *If n is odd, then $S = g^{\frac{n-5}{2}}(\frac{n+3}{2}g)^2(\frac{n-1}{2}g)$ when $n \geq 9$ and $S = g \cdot (3g)^2 \cdot (4g) \cdot (7g)$ when $n = 9$. Moreover $\text{ind}(S) = 2$.*
- (2) *If n is even, then either $S = (2g)^{\frac{n}{2}-1}(x_1g)((n+2-x_1)g)$, where $2 \nmid x_1, 1 < x_1 < n$, $x_1 \neq n+2-x_1$ or $S = g^t(\frac{n}{2}g)((1+\frac{n}{2})g)^{2\ell}$, where t, ℓ are positive integers with $t+2\ell = \frac{n}{2}$. Moreover $\text{ind}(S) \geq 2$.*

In this paper, we characterized the unsplittable minimal zero-sum sequences of length $|S| = \mathsf{l}(G) - 2$ over a cyclic group G of prime order. Our main results state as following.

Theorem 1.4. *Let $p > 155$ be a prime and let G be a cyclic group of order p . Let S be an unsplittable minimal zero-sum sequence of length $|S| = \frac{p-1}{2}$ over G . We have S is one of the following forms:*

$$g^{\frac{p-11}{2}}(\frac{p+3}{2}g)^4(\frac{p-1}{2}g) \text{ or } g^{\frac{p-7}{2}}(\frac{p+5}{2}g)^2(\frac{p-3}{2}g).$$

Moreover $\text{ind}(S) = 2$.

Theorem 1.5. *Let $p > 155$ be a prime and let G be a cyclic group of order p . Let T be a minimal zero-sum sequence of length $|T| \geq \mathsf{l}(G) - 2 = \frac{p-1}{2}$ over G . We have $\text{ind}(T) \leq 2$.*

The paper is organized as follows. In the next section, we provide some preliminary results. In section 3, we give a proof for our main results. In the last section, we will give some further remarks.

2. PRELIMINARIES

Our notation and terminology are consistent with [6] and [11]. Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and for real numbers a, b let $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$.

Let G be an additive finite abelian group. Every sequence S over G can be written in the form

$$S = g_1 \cdot \dots \cdot g_\ell = \prod_{g \in G} g^{\mathbf{v}_g(S)}, \quad \text{with } \mathbf{v}_g(S) \in \mathbb{N}_0 \text{ for all } g \in G.$$

where $\mathbf{v}_g(S) \in \mathbb{N}_0$ denote the *multiplicity* of g in S . We call

$$\begin{aligned} \text{supp}(S) &= \{g \in G \mid \mathbf{v}_g(S) > 0\} \text{ the } \textit{support} \text{ of } S; \\ \mathbf{h}(S) &= \max\{\mathbf{v}_g(S) \mid g \in G\} \text{ the } \textit{maximum of the multiplicities} \text{ of } g \text{ in } S; \\ |S| = \ell &= \sum_{g \in G} \mathbf{v}_g(S) \in \mathbb{N}_0 \text{ be the } \textit{length} \text{ of } S; \\ \sigma(S) &= \sum_{i=1}^{\ell} g_i = \sum_{g \in G} \mathbf{v}_g(S)g \in G \text{ be the } \textit{sum} \text{ of } S. \end{aligned}$$

A sequence T is called a *subsequence* of S and denoted by $T \mid S$ if $\mathbf{v}_g(T) \leq \mathbf{v}_g(S)$ for all $g \in G$. Whenever $T \mid S$, let ST^{-1} denote the subsequence with T deleted from S . If S_1, S_2 are two disjoint subsequences of S , let

$$S_1 S_2$$

denote the subsequence of S satisfying that $\mathbf{v}_g(S_1 S_2) = \mathbf{v}_g(S_1) + \mathbf{v}_g(S_2)$ for all $g \in G$. Let

$$\Sigma(S) = \{\sigma(T) \mid T \text{ is a subsequence of } S \text{ with } 1 \leq |T| \leq |S|\}.$$

The sequence S is called

$$\begin{aligned} \textit{zero-sum} & \quad \text{if } \sigma(S) = 0 \in G; \\ \textit{zero-sum free} & \quad \text{if } 0 \notin \Sigma(S); \\ \textit{minimal zero-sum} & \quad \text{if } \sigma(S) = 0 \text{ and } \sigma(T) \neq 0 \text{ for every } T \mid S \text{ with } 1 \leq |T| < |S|. \end{aligned}$$

Lemma 2.1. [13, Theorem 5.3.1] *Let G be an abelian group. Let S be a zero-sum free sequence over G . Suppose $S = S_1 S_2 \cdots S_t$, then $|\Sigma(S)| \geq \sum_{i=1}^t (|\Sigma(S_i)|)$.*

Lemma 2.2. [1] *Let p be a prime and let G be a cyclic group of order p . Suppose $A \subset G$ and $A \cap (-A) = \emptyset$. Then $|\Sigma(A)| \geq \min\{p, \frac{|A|(|A|+1)}{2}\}$.*

Lemma 2.3. *Let p be a prime and let G be a cyclic group of order p . Let A be a zero-sum free subset of G , then $|\Sigma(A)| \geq \min\{p, \frac{|A|(|A|+1)}{2}\}$.*

Proof. Since A is a zero-sum free subset, we have $A \cap (-A) = \emptyset$. Hence the results follows from Lemma 2.2. \square

Lemma 2.4. [26, Lemma 2.14] *Let p be a prime and let G be a cyclic group of order p . Suppose S is a minimal zero-sum sequence of elements over G . Then S is unsplittable if and only if $|\Sigma(Sg^{-1})| = p - 1$ for every $g \in \text{supp}(S)$.*

Lemma 2.5. [26, Lemma 2.15] *Let p be a prime and let G be a cyclic group of order p . Let S be a minimal zero-sum sequence consisting of two distinct elements. Then S is splittable.*

For convenience, from Lemma 2.6 till Lemma 2.10 we always assume that p is a prime and G is a cyclic group of order p . Let S be an unsplittable minimal zero-sum sequence of elements over G .

Lemma 2.6. [26, Lemma 2.5] *Suppose $g, tg \in \text{supp}(S)$ with $t \in [2, p-1]$. Then $t \geq \mathbf{v}_g(S) + 2$. Moreover $t \neq \frac{p+1}{2}$.*

Lemma 2.7. [26, Lemma 2.6] *Suppose $g, h \in \text{supp}(S)$ with $g \neq h$. Then*

- (1) *If $k \in [0, \mathbf{v}_g(S)]$, then $|\Sigma(g^k h)| = 2k + 1$.*
- (2) *If $\mathbf{v}_g(S) \geq 2$ and $\mathbf{v}_h(S) \geq 2$, then $|\Sigma(g^2 h^2)| = 8$.*

Lemma 2.8. *Let $T = g^k(xg)^2$ be a subsequence of S , where $k \geq 3$. Then $|\Sigma(T)| \geq 2|T|$. Moreover apart from the case $T = g^k(\frac{p+3}{2}g)^2$, $|\Sigma(T)| \geq 2|T| + 1$.*

Proof. Since S is unsplittable, by Lemmas 2.6, we have $x \geq k + 2$ and $x \neq \frac{p+1}{2}$.

If $2x < p$, since S has minimal zero-sum, we have $2x + k < p$. Then $g, 2g, \dots, kg, xg, (x+1)g, \dots, (x+k)g, 2xg, (2x+1)g, \dots, (2x+k)g$ are pairwise distinct and hence $|\Sigma(T)| \geq 3k + 2 \geq 2|T| + 1$.

Next assume that $2x > p$. Then $x \geq \frac{p+3}{2}$. Since S has minimal zero-sum, we have $x + k < p$, and hence $x > 2x - p + k$.

If $2x - p > k$, then $g, 2g, \dots, kg, (2x - p)g, (2x - p + 1)g, \dots, (2x - p + k)g, xg, (x+1)g, \dots, (x+k)g$ are pairwise distinct and hence $|\Sigma(T)| \geq 3k + 2 \geq 2|T| + 1$.

If $2x - p \leq k$, then $g, 2g, \dots, kg, (k+1)g, \dots, (2x - p + k)g, xg, (x+1)g, \dots, (x+k)g$ are pairwise distinct and hence $|\Sigma(T)| \geq (2x - p + k) + (k+1) \geq 2|T|$, and the equality holds if and only if $x = \frac{p+3}{2}$. \square

Lemma 2.9. [26, Lemma 2.11] *Let $T = g_1^k g_2 g_3$ be a subsequence of S . Then $|\Sigma(T)| \geq 2|T|$, moreover apart from the case $T = g_1^k(\frac{p-1}{2}g_1)(\frac{p+3}{2}g_1)$, $|\Sigma(T)| \geq 2|T| + 1$.*

Lemma 2.10. *Let T be a subsequence of S . If $|\text{supp}(T)| \geq 2$, then there exists $g \in \text{supp}(T)$ such that $|\Sigma(g^{-1}T)| \geq 2|g^{-1}T| - 1$.*

Proof. Since $|\text{supp}(T)| \geq 2$, we can write

$$T = U_1 \cdot \dots \cdot U_t V_1 \cdot \dots \cdot V_r W,$$

where U_1, U_2, \dots, U_t are 3-subsets of G , V_1, V_2, \dots, V_r are of form $g^2 h^2$ with $g, h \in \text{supp}(T)$ and $W = g^x h^y$ with $y \leq 1$. By Lemma 2.3 we have $|\Sigma(U_i)| \geq 6 = 2|U_i|$ for $i = 1, 2, \dots, t$. By Lemma 2.7.2 we have $|\Sigma(V_j)| = 8 = 2|V_j|$ for $j = 1, 2, \dots, r$.

If $y = 1$, then by Lemma 2.7.1 we have $|\Sigma(g^{-1}W)| \geq 2|g^{-1}W| - 1$. By Lemma 2.1, we infer that $|\Sigma(Tg^{-1})| \geq \sum_{i=1}^t |\Sigma(U_i)| + \sum_{j=1}^r |\Sigma(V_j)| + |\Sigma(g^{-1}W)| \geq 2\sum_{i=1}^t |U_i| + 2\sum_{j=1}^r |V_j| + 2|Wg^{-1}| - 1 = 2|Tg^{-1}| - 1$, and we are done.

If $y = 0$, we have that $t \geq 1$ or $r \geq 1$. If $t \geq 1$, in view of Lemmas 2.3 and 2.9, there exists $g \in \text{supp}(T)$ such that $|\Sigma(WU_t g^{-1})| \geq 2|WU_t g^{-1}| - 1$. Therefore by Lemma 2.1, we infer that $|\Sigma(Tg^{-1})| \geq 2|Tg^{-1}| - 1$, and we are done. If $r \geq 1$, then by Lemma 2.7.1, we have $|\Sigma(WV_r h^{-1})| \geq 2|WV_r h^{-1}| - 1$. Also by Lemma 2.1, we infer that $|\Sigma(Th^{-1})| \geq 2|Th^{-1}| - 1$, and we are done.

This completes the proof. \square

3. PROOF OF THE MAIN RESULTS

Throughout this section, we always assume that

- (1) $p > 155$ is a prime;
- (2) G is a cyclic group of order p ;
- (3) S is an unsplittable minimal zero-sum sequence of length $\frac{p-1}{2}$ over G .

Lemma 3.1. $3 \leq |\text{supp}(S)| \leq 4$.

Proof. Since S is unsplittable, by Lemma 2.5, we have $|\text{supp}(S)| \geq 3$. It remains to show that $|\text{supp}(S)| \leq 4$.

Assume to the contrary that $|\text{supp}(S)| \geq 5$. Suppose $S = g_1^{r_1} g_2^{r_2} \cdots g_k^{r_k}$, where $r_1 \geq r_2 \geq \cdots \geq r_k \geq 1$ and $k \geq 5$. We can write

$$S = TU,$$

where $T = g_1 g_2 \cdots g_5$ and $|U| = \frac{p-1}{2} - 5 = \frac{p-11}{2}$. By Lemma 2.3 we have $|\Sigma(T)| \geq 15$.

If $|\text{supp}(U)| \geq 2$, by Lemma 2.10, there exists $a \in [1, k]$ such that $|\Sigma(Ug_a^{-1})| \geq 2|Ug_a^{-1}| - 1$. By Lemma 2.1, we infer that $|\Sigma(Sg_a^{-1})| \geq |\Sigma(T)| + |\Sigma(Ug_a^{-1})| \geq 15 + 2|Ug_a^{-1}| - 1 = 15 + 2(\frac{p-11}{2} - 1) - 1 > p$, yielding a contradiction to Lemma 2.4.

Next assume that $|\text{supp}(U)| = 1$. Then $k = 5$ and $U = g_1^{r_1-1}$. Hence we can write

$$S = g^{r_1}(t_2 g) \cdots (t_5 g)$$

with $2 \leq t_2 < \cdots < t_5 \leq p-1$. Then $r_1 = \frac{p-1}{2} - (5-1) = \frac{p-9}{2}$. By Lemma 2.6, we have $t_2 \geq r_1 + 2 = \frac{p-5}{2}$. Since S has minimal zero-sum, we have $t_5 \leq p - r_1 - 1 = \frac{p+7}{2}$. Since $p \geq 19$ we infer that $r_1 + t_2 + t_3 + t_4 + t_5 \not\equiv 0 \pmod{p}$, yielding a contradiction to that S is zero-sum.

Therefore $|\text{supp}(S)| \leq 4$. This completes the proof. \square

Lemma 3.2. Suppose $S = g^{r_1}(t_2 g)^{r_2}(t_3 g)^{r_3}(t_4 g)^{r_4}$, where $2 \leq t_2, t_3, t_4 \leq p-1$ and $r_2 + r_3 + r_4 \leq 15$. If $r_i \geq 2$ for some $i \in \{2, 3, 4\}$, then $t_i \geq \frac{p+3}{2}$.

Proof. Since $r_2 + r_3 + r_4 \leq 15$, we have $r_1 = |S| - r_2 - r_3 - r_4 \geq \frac{p-31}{2}$. Since S is unsplittable, by Lemma 2.6 we infer that $t_2, t_3, t_4 \geq r_1 + 2 \geq \frac{p-27}{2}$ and $t_2, t_3, t_4 \neq \frac{p+1}{2}$. Since S has minimal zero-sum, $t_2, t_3, t_4 \leq p - r_1 - 1 \leq \frac{p+29}{2}$. Then $p - 27 \leq 2t_2, 2t_3, 2t_4 \leq p + 29$.

Next assume that $r_i \geq 2$ for some $i \in \{2, 3, 4\}$. If $2t_i < p$, since S is a minimal zero-sum sequence, we infer that $2t_i \leq p - r_1 - 1 \leq \frac{p+29}{2}$, which implies $p \leq 83$, yielding a contradiction. Hence $2t_i > p$. Moreover $t_i \geq \frac{p+3}{2}$. \square

Lemma 3.3. $\text{supp}(S) = 3$.

Proof. By Lemma 3.1, we have $|\text{supp}(S)| \in [3, 4]$. Assume that $|\text{supp}(S)| = 4$ and $S = g_1^{r_1} g_2^{r_2} g_3^{r_3} g_4^{r_4}$, where $r_1 \geq r_2 \geq r_3 \geq r_4 > 0$.

By Lemma 2.8, we have that either $|\Sigma(g_i^2 g_j^2)| \geq 11$ or $|\Sigma(g_i^2 g_j^3)| \geq 11$ for $i, j \in \{1, 2, 3\}$. By Lemma 2.3 we have $|\Sigma(g_1 g_2 g_3 g_4)| \geq 10$.

We first show that $r_4 = 1$. Assume to the contrary that $r_4 \geq 2$. Write

$$S = T_1 \cdots T_{r_4} U,$$

where $T_1 = \cdots = T_{r_4} = g_1 g_2 g_3 g_4$, $|\text{supp}(U)| \leq 3$. If $|\text{supp}(U)| \geq 2$, by Lemma 2.10, there exist $a \in \{1, 2, 3\}$ such that $|\Sigma(g_a^{-1}U)| \geq 2|g_a^{-1}U| - 1$. Then by Lemma 2.1 we infer that $|\Sigma(Sg_a^{-1})| \geq \sum_{i=1}^{r_4} |\Sigma(T_i)| + |\Sigma(Ug_a^{-1})| \geq 2(|S| - 1) + 2r_4 - 1 \geq p$, yielding a contradiction to Lemma 2.4. Hence we may assume that $r_2 = r_3 = r_4$ and therefore $U = g_1^{r_1 - r_4}$. If $r_4 \geq 2$, then by Lemma 2.9, there exists $a \in \{2, 3, 4\}$ such that $|\Sigma(T_{r_4}Ug_a^{-1})| \geq 2|T_{r_4}Ug_a^{-1}| + 1$. By Lemma 2.1, we infer that $|\Sigma(Sg_a^{-1})| \geq \sum_{i=1}^{r_4-1} |\Sigma(T_i)| + |\Sigma(T_{r_4}Ug_a^{-1})| \geq 2(|S| - 1) + 2(r_4 - 1) + 1 \geq p$, yielding a contradiction. Therefore $r_4 = 1$.

Second we will show that $r_2 \leq 7$. Assume to the contrary that $r_2 \geq 8$. Write

$$S = TU_1U_2V,$$

where $T = g_1 g_2 g_3 g_4$, $U_1 = U_2 = g_1^3 g_2^2$ or $g_1^2 g_2^3$ such that $|\Sigma(U_i)| \geq 11 = 2|U_i| + 1$ for $i = 1, 2$, $|\text{supp}(V)| \geq 2$. In view of Lemmas 2.3 and 2.10, there exists $a \in \{1, 2, 3\}$ such that $|\Sigma(g_a^{-1}V)| \geq 2|g_a^{-1}V| - 1$. By Lemma 2.1, we infer that $|\Sigma(Sg_a^{-1})| \geq |\Sigma(T)| + \sum_{i=1}^2 |\Sigma(U_i)| + |\Sigma(g_a^{-1}V)| \geq 2(|S| - 1) + 4 - 1 \geq p$, yielding a contradiction to Lemma 2.4. Hence $r_2 \leq 7$.

Next we will show that $r_3 = 1$. Assume to the contrary that $r_3 \geq 2$. By Lemma 3.2, we infer that $t_2, t_3 \geq \frac{p+3}{2}$. Since $p > 155$, we have that $r_1 > 56$.

If $r_3 \geq 4$, write

$$S = TU_1U_2V,$$

where $T = g_1 g_2 g_3 g_4$, $U_1 = g_1^3 g_2^2$ or $g_1^2 g_2^3$, $U_2 = g_1^3 g_3^2$ or $g_1^2 g_3^3$ such that $|\Sigma(U_i)| \geq 11 = 2|U_i| + 1$ for $i = 1, 2$, $|\text{supp}(V)| \geq 2$. In view of Lemmas 2.3 and 2.10, there exists $a \in \{1, 2, 3\}$ such that $|\Sigma(g_a^{-1}V)| \geq 2|g_a^{-1}V| - 1$. By Lemma 2.1, we infer that $|\Sigma(Sg_a^{-1})| \geq |\Sigma(T)| + \sum_{i=1}^2 |\Sigma(U_i)| + |\Sigma(g_a^{-1}V)| \geq 2(|S| - 1) + 4 - 1 \geq p$, yielding a contradiction to Lemma 2.4.

If $r_3 \leq 3$, then $r_1 = |S| - r_2 - r_3 - 1 \geq \frac{p-23}{2}$. Since S is a minimal zero-sum sequence, we infer that $t_i \leq p - r_1 - 1 \leq \frac{p+21}{2}$ for $i = 2, 3, 4$. Since $p > 155$, we infer that $g^{r_1}(t_2 g)^5(t_3 g)^2$ contains a zero-sum subsequence. This together with S is a minimal zero-sum sequence forces that $r_2 \leq 4$. Hence $r_1 = |S| - r_2 - r_3 - 1 \geq \frac{p-17}{2}$. Similarly since $g^{r_1}(t_2 g)^3(t_3 g)^2$ contains a zero-sum subsequence, we infer that $r_2 = r_3 = 2$. Hence $r_1 = |S| - r_2 - r_3 - 1 = \frac{p-11}{2}$. But $g^{r_1}(t_2 g)^2(t_3 g)^2$ contains a zero-sum subsequence, yielding a contradiction to that \bar{S} is a minimal zero-sum sequence.

Therefore $r_3 = 1$.

Since $r_1 = |S| - r_2 - r_3 - r_4 = \frac{p-5-2r_2}{2}$, by Lemma 2.6, $t_i \geq r_1 + 2 = \frac{p-1-2r_2}{2}$ for $i = 2, 3, 4$. Since S is a minimal zero-sum sequence, we have $t_i \leq p - r_1 - 1 = \frac{p+3+2r_2}{2}$ for $i = 2, 3, 4$. Now assume that $t_i = \frac{p+x_i}{2}$, then $-1 - 2r_2 \leq x_i \leq 2r_2 + 3$ for $i = 2, 3, 4$.

Since S is a zero-sum sequence, we have

$$r_1 + t_2 r_2 + t_3 r_3 + t_4 r_4 = \frac{p-5-2r_2}{2} + \frac{p+x_2}{2} r_2 + \frac{p+x_3}{2} + \frac{p+x_4}{2} \equiv 0 \pmod{p}.$$

Since p is odd prime, we have $p - 5 - 2r_2 + p r_2 + x_2 r_2 + p + x_3 + p + x_4 \equiv 0 \pmod{p}$. Hence

$$(x_2 - 2)r_2 + x_3 + x_4 - 5 \equiv 0 \pmod{p}.$$

Recalling that S is a minimal zero-sum sequence, $p > 155$, $r_2 \leq 7$, and $-1 - 2r_2 \leq x_i \leq 2r_2 + 3$ for $i = 2, 3, 4$, it is easy to check that

$$r_2 = 1, \text{ and } \{t_2, t_3, t_4\} = \{\frac{p-1}{2}, \frac{p+3}{2}, \frac{p+5}{2}\}.$$

Therefore $S = g^{\frac{p-7}{2}}(\frac{p-1}{2}g)(\frac{p+3}{2}g)(\frac{p+5}{2}g)$. But $|\Sigma(S(\frac{p+5}{2}g)^{-1})| = p - 3$, yielding a contradiction to Lemma 2.4.

Hence $|\text{supp}(S)| = 3$. This completes the proof. \square

Lemma 3.4. S is not of form $S = g^{r_1}(\frac{p-1}{2}g)^{r_2}(\frac{p+3}{2}g)^{r_3}$ with $r_1 \geq r_2 \geq r_3 > 0$.

Proof. Assume to the contrary that such S exists. Since $|S| = r_1 + r_2 + r_3 = \frac{p-1}{2}$ and $r_1 \geq r_2 \geq r_3$, we infer that $r_2 \leq \frac{p-1}{4}$. Since S is a zero-sum sequence, we have that $r_1 + \frac{p-1}{2}r_2 + \frac{p+3}{2}r_3 \equiv 0 \pmod{p}$. Hence

$$2r_1 - r_2 + 3r_3 \equiv 0 \pmod{p}.$$

This together with $r_1 + r_2 + r_3 = \frac{p-1}{2}$ gives that $3r_2 - r_3 \equiv p - 1 \pmod{p}$. Which is impossible since $r_2 \geq r_3$ and $r_2 \leq \frac{p-1}{4}$. \square

Lemma 3.5. Suppose $S = g^{r_1}(t_2g)^{r_2}(t_3g)^{r_3}$, where $r_1 \geq r_2 \geq r_3 > 0$. Then $S = g^{\frac{p-11}{2}}(\frac{p+3}{2}g)^4(\frac{p-1}{2}g)$ or $g^{\frac{p-7}{2}}(\frac{p+5}{2}g)^2(\frac{p-3}{2}g)$.

Proof. By Lemma 2.4, if $S = g^{\frac{p-11}{2}}(\frac{p+3}{2}g)^4(\frac{p-1}{2}g)$ or $g^{\frac{p-7}{2}}(\frac{p+5}{2}g)^2(\frac{p-3}{2}g)$, it is easy to check that S is unsplitable. It remains to show that S is of above forms.

Since S is a minimal zero-sum sequence, in view of Lemma 2.6, we obtained that $r_2 \geq 2$.

Case 1. $t_2 \neq \frac{p+3}{2}$. By Lemma 2.8, $|\Sigma(g^3(t_2g)^2)| \geq 11$. By Lemmas 3.4 and 2.9, we infer that $|\Sigma(g^k(t_2g)(t_3g))| \geq 2(k+2) + 1$.

Now write

$$S = T_1 \cdots T_x U_1 \cdots U_y V,$$

where $T_1 = \cdots = T_x = g^2(t_2g)(t_3g)$, $U_1 = \cdots = U_y = g^3(t_2g)^2$ and $|\text{supp}(V)| \leq 2$. Clearly $x \geq 1$.

If $x + y \geq 5$, then $|\text{supp}(T_x V)| \geq 3$. By Lemma 2.10, there exists $a \in \{1, t_2, t_3\}$ such that $|\Sigma((ag)^{-1}T_x V)| \geq 2|(ag)^{-1}T_x V| - 1$. By Lemma 2.1, we infer that $|\Sigma(S(ag)^{-1})| \geq \sum_{i=1}^{x-1} |\Sigma(T_i)| + \sum_{j=1}^y |\Sigma(U_j)| + |\Sigma((ag)^{-1}T_x V)| \geq 2(|S| - 1) + (x + y - 1) - 1 \geq p$, yielding a contradiction to Lemma 2.4.

If $x + y = 4$, since $p > 155$, we infer that $|\text{supp}(V)| \geq 1$. If $|\text{supp}(V)| = 2$, by Lemma 2.10, there exists $a \in \{1, t_2, t_3\}$ such that $|\Sigma((ag)^{-1}V)| \geq 2|(ag)^{-1}V| - 1$. By Lemma 2.1, we infer that $|\Sigma(S(ag)^{-1})| \geq \sum_{i=1}^x |\Sigma(T_i)| + \sum_{j=1}^y |\Sigma(U_j)| + |\Sigma((ag)^{-1}V)| \geq 2(|S| - 1) + (x + y) - 1 \geq p$, yielding a contradiction. If $|\text{supp}(V)| = 1$, we infer that $V = g^k$. Then by Lemma 2.9, $|\Sigma(g^{-1}T_x V)| \geq 2|g^{-1}T_x V|$. By Lemma 2.1, we infer that $|\Sigma(Sg^{-1})| \geq \sum_{i=1}^{x-1} |\Sigma(T_i)| + \sum_{j=1}^y |\Sigma(U_j)| + |\Sigma(g^{-1}T_x V)| \geq 2(|S| - 1) + (x + y - 1) \geq p$, yielding a contradiction.

Therefore

$$x + y \leq 3.$$

Then $r_3 \leq 3$ and moreover $r_2 + r_3 \leq 7$. Since $r_2 \geq 2$, by Lemma 3.2, we have $t_2 \geq \frac{p+5}{2}$. Note that $r_1 = |S| - r_2 - r_3 \geq \frac{p-15}{2}$. Since S is a minimal zero-sum sequence, $t_2, t_3 \leq p - r_1 - 1 \leq \frac{p+13}{2}$. Since $p > 155$, we have $g^{r_1}(t_2g)^3$ contains a zero-sum subsequence, yielding a contradiction. Hence we may assume that $r_2 = 2$.

If $r_3 = 2$, then $r_1 = |S| - r_2 - r_3 = \frac{p-9}{2}$. Since S is a minimal zero-sum sequence, $t_2, t_3 \leq p - r_1 - 1 = \frac{p+7}{2}$. By Lemma 3.2, $t_3 \geq \frac{p+3}{2}$. Since $p > 155$, we have $g^{r_1}(t_2g)(t_3g)^2$ contains a zero-sum subsequence, yielding a contradiction. Hence we may assume that $r_3 = 1$. Then $r_1 = |S| - r_2 - r_3 = \frac{p-7}{2}$. Since S is a minimal zero-sum sequence, we have $t_2 \leq p - r_1 - 1 = \frac{p+5}{2}$. Therefore $t_2 = \frac{p+5}{2}$. Then $t_3 = \frac{p-3}{2}$ and hence $S = g^{\frac{p-7}{2}}(\frac{p+5}{2}g)^2(\frac{p-3}{2}g)$.

Case 2. $t_2 = \frac{p+3}{2}$. By Lemma 2.8, $|\Sigma(g^2(t_2g)^3)| \geq 11$ and $|\Sigma(g^3(t_3g)^2)| \geq 11$.

We first show that $r_2 \leq 11$ and $r_3 \leq 5$.

If $r_2 \geq 12$, then we can write

$$S = T_1 T_2 T_3 T_4 U,$$

where $T_1 = T_2 = T_3 = T_4 = g^2(t_2g)^3$ and $|\text{supp}(U)| \geq 2$. By Lemma 2.10, there exists $a \in \{1, t_2, t_3\}$ such that $|\Sigma(U(ag)^{-1})| \geq 2|U(ag)^{-1}| - 1$. By Lemma 2.1, we infer that $|\Sigma(S(ag)^{-1})| \geq \sum_{i=1}^4 |\Sigma(T_i)| + |\Sigma(U(ag)^{-1})| \geq 2(|S| - 1) + 4 - 1 \geq p$, yielding a contradiction to Lemma 2.4. Hence we may assume that $r_2 \leq 11$. Since $p \geq 100$, then $r_1 \geq 11$. If $r_2 \geq r_3 \geq 6$, then we can write

$$S = T_1 T_2 U_1 U_2 V,$$

where $T_1 = T_2 = g^2(t_2g)^3$, $U_1 = U_2 = g^3(t_3g)^2$ and $|\text{supp}(V)| \geq 2$. By Lemma 2.10, there exists $a \in \{1, t_2, t_3\}$ such that $|\Sigma(V(ag)^{-1})| \geq 2|V(ag)^{-1}| - 1$. Also by Lemma 2.1, we infer that $|\Sigma(S(ag)^{-1})| \geq p$, yielding a contradiction. Hence $r_3 \leq 5$.

Since $r_1 = |S| - r_2 - r_3 = \frac{p-1-2r_2-2r_3}{2}$, by Lemma 2.6, $t_3 \geq r_1 + 2 = \frac{p+3-2r_2-2r_3}{2}$. Since S is a minimal zero-sum sequence, we have $t_3 \leq p - r_1 - 1 = \frac{p-1+2r_2+2r_3}{2}$. Now assume that $t_3 = \frac{p+x}{2}$, then $3 - 2r_2 - 2r_3 \leq x \leq 2r_2 + 2r_3 - 1$. Hence $-29 \leq x \leq 31$.

Since S is a zero-sum sequence, we have

$$r_1 + t_2 r_2 + t_3 r_3 = \frac{p-1-2r_2-2r_3}{2} + \frac{p+3}{2} r_2 + \frac{p+x}{2} r_3 \equiv 0 \pmod{p}.$$

Since p is an odd prime, we have $p-1-2r_2-2r_3+pr_2+3r_2+pr_3+xr_3 \equiv 0 \pmod{p}$. Hence

$$r_2 + (x-2)r_3 - 1 \equiv 0 \pmod{p}.$$

Recalling that S is a minimal zero-sum sequence, $p > 155$, $r_2 \leq 11$, $r_3 \leq 5$ and $-29 \leq x \leq 31$, it is easy to check that

$$r_2 = 4, r_3 = 1 \text{ and } t_3 = \frac{p-1}{2}.$$

We are done. This completes the proof. \square

Lemma 3.6. Suppose $g \in G \setminus \{0\}$ and S if one of the following forms

$$g^{\frac{p-11}{2}}(\frac{p+3}{2}g)^4(\frac{p-1}{2}g) \text{ or } g^{\frac{p-7}{2}}(\frac{p+5}{2}g)^2(\frac{p-3}{2}g).$$

Then $\text{ind}(S) = 2$.

Proof. Suppose $h \in G \setminus \{0\}$, then $g = mh$ for some $m \in [1, p-1]$. If $S = g^{\frac{p-11}{2}}(\frac{p+3}{2}g)^4(\frac{p-1}{2}g)$, it is easy to check that $\|S\|_h \geq 2$ and if $g = 2h$, then $\|S\|_h = 2$. Hence $\text{ind}(S) = 2$. Similarly, we can show that if $S = g^{\frac{p-7}{2}}(\frac{p+5}{2}g)^2(\frac{p-3}{2}g)$, then $\text{ind}(S) = 2$. \square

Now we are in a position to proof the main results.

Proof of Theorem 1.4: Suppose S is an unsplittable minimal zero-sum sequence of length $\frac{p-1}{2}$. By Lemma 3.3, we have $|\text{supp}(S)| = 3$. Then by Lemma 3.5, $S = g^{\frac{p-11}{2}}(\frac{p+3}{2}g)^4(\frac{p-1}{2}g)$ or $g^{\frac{p-7}{2}}(\frac{p+5}{2}g)^2(\frac{p-3}{2}g)$. By Lemma 3.6, we have $\text{ind}(S) = 2$. This completes the proof. \square

Proof of Theorem 1.5: If $|T| \geq \frac{p+3}{2}$, by Theorem 1.2, $\text{ind}(T) = 1$.

Next assume that $|T| = \frac{p+1}{2}$. If T is unsplittable, by Theorem 1.3.1, we have $\text{ind}(T) = 2$. If T is splittable, i.e., there exists $h \in \text{supp}(T)$ and $x, y \in G$ such that $h = x + y$ and $T' = xyTh^{-1}$ is also a minimal zero-sum sequence of length $\frac{p+3}{2}$. Then by Theorem 1.2, $\text{ind}(T') = 1$. Clearly $\|T\|_g \leq \|T'\|_g$ for every $g \in G \setminus \{0\}$. Hence $\text{ind}(T) \leq \text{ind}(T') = 1$.

If $|T| = \frac{p-1}{2}$, similar to above we can show that $\text{ind}(T) \leq 2$. This completes the proof. \square

4. CONCLUDING REMARKS

Let p be a prime and let G be a cyclic group of order p . When $p < 155$, it is not hard to characterize the structure of unsplittable minimal zero-sum sequence of length $|S| = \frac{p-1}{2}$. Similar to Theorems 1.4 and 1.5, we can show that

Theorem 4.1. *Let $p > 200$ be a prime and let G be a cyclic group of order p . Let S be an unsplittable minimal zero-sum sequence of length $|S| = \frac{p-3}{2}$ over G . We have S is one of the following forms:*

$$g^{\frac{p-17}{2}}(\frac{p+3}{2}g)^6(\frac{p-1}{2}g) \text{ or } g^{\frac{p-9}{2}}(\frac{p+7}{2}g)^2(\frac{p-5}{2}g).$$

Theorem 4.2. *Let $p > 200$ be a prime and let G be a cyclic group of order p . Let T be a minimal zero-sum sequence of length $|T| \geq \text{l}(G) - 3 = \frac{p-3}{2}$ over G . We have $\text{ind}(T) \leq 2$.*

Definition 4.3.

1. Let n be an integer. $\text{l}(n)$ denotes the maximal value of index of minimal zero-sum sequences S over a cyclic group G of order n .
2. Let G be a finite cyclic group and $k \geq 1$ be an integer. $\text{l}_k(G)$ denotes the smallest integer $l \in \mathbb{N}$ such that every minimal zero-sum sequence S of length $|S| \geq l$ has $\text{ind}(S) \leq k$.

To determine $\text{l}(n)$ is proposed by Gao [5], and he conjectured that $\text{l}(n) \leq c \ln n$ for some absolute constant c [5, Conjecture 4.2]. If $n \equiv 0 \pmod{8}$, let G be a cyclic group of order n . Suppose

$$S = g^{\frac{n}{4}}(\frac{n}{2}g)((1 + \frac{n}{2})g)^{\frac{n}{4}}.$$

Then $\text{ind}(S) = \frac{n}{8} + 1$. Hence the conjecture of Gao is not true for $n \equiv 0 \pmod{8}$. In fact, the conjecture is also not true for every even n (see Theorem 1.3.2).

Let G be a finite cyclic group of order n . Clearly, if $k \geq l(n)$, then $l_k(G) = 1$. If $k = 1$, then $l_1(G) = l(G)$. By Theorem 4.2, we infer that $l_2(G) \leq \frac{p-3}{2}$, provided that $n = p$ is prime.

Problem. Determine $l(n)$ for all integers n and determine $l_k(G)$ for all the cyclic groups G .

Acknowledgements. This work has been supported by the National Science Foundation of China (Grant Nos. 11271207 and 11301531) and a research grant from Civil Aviation University of China (No. 2010QD02X).

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